

Analytical Solution For Navier-Stokes Equations In Two Dimensions For Laminar Incompressible Flow

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Abstract

The Navier-Stokes equations describing laminar flow of an incompressible fluid will be solved. Different group of general solutions for Navier stokes equations governing Laminar incompressible fluids will be derived.

Keywords: Applied mathematics , Fluid dynamics, Navier-Stokes equations

1 Introduction

Navier stokes equations for laminar flows, in different physical conditions, have been solved in almost all fluid mechanics text books(1). In some physical conditions these equations have exact solutions, but in general these equations are complicated nonlinear partial differential equations that can not be solved easily. Therefore, the authors in most cases, in order to solve these equations, have either used numerical methods, or they have appealed to Physically acceptable simplifying considerations. In this way so many efforts have been put on the recognition of physical parameters and terms that could be neglected without affecting the system effectively.

Also, Navier stokes equations have a very wide application in stellar astrophysics,[2],[3],[4].

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Since 2000, The author has tried to somehow overcome the complexities governing the analytical solutions of nonlinear partial Differential equations and in many respects these activities has been successful.[5],[6],[7],[8],[9],[10]. Since Navier stokes equations governing the incompressible fluids has a wide application in technology and industry, in this article we try to find explicit solutions of these equations for the laminar incompressible fluids in two dimensions. We have treated these set of equations in a general way without restricting ourselves to any especial boundary value conditions, although from the results found in sections 2 and 3, one can conclude that these results are consistent with different, logical and acceptable, boundary conditions. How to fit the results with a especial set of boundary conditions is beyond the scope of this article. We will address this issue in our next efforts.

2 Basic equations

Navier stokes equations in two dimensions in cartesian coordinates for laminar incompressible fluids are;

$$u(x, y) \frac{\partial u(x, y)}{\partial x} + v(x, y) \frac{\partial u(x, y)}{\partial y} = -\frac{1}{\rho} \frac{\partial}{\partial x} P(x, y) + \nu \left(\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} \right) \quad (1)$$

$$u(x, y) \frac{\partial v(x, y)}{\partial x} + v(x, y) \frac{\partial v(x, y)}{\partial y} = -\frac{1}{\rho} \frac{\partial}{\partial y} P(x, y) + \nu \left(\frac{\partial^2 v(x, y)}{\partial x^2} + \frac{\partial^2 v(x, y)}{\partial y^2} \right) \quad (2)$$

$$\frac{\partial u(x, y)}{\partial x} + \frac{\partial v(x, y)}{\partial y} = 0 \quad (3)$$

Here, $u(x, y)$ and $v(x, y)$ are the x and y components of the velocity respectively, and $P(x, y)$ stands for the pressure. The two first equations are the components of momentum equations, in the x and y direction. and obviously the third equation is the continuity equation.

To solve the above set of equations, at first we suppose that, u and v , are functions of a float function $f(x, y)$, that is to say,

$$u = u(f(x, y)), v = v(f(x, y)) \quad (4)$$

However, this is a restricting assumption, it has no contradiction with logical considerations based on the following discussions.

1-What we do is similar to the application of separation method in solving the problems. At the beginning, when we apply this method we do not have any logical supportive argument behind that. We hope by writing the function in the separated form, we will come to suitable results. Of course, It is possible that it may not work and in fact in many cases it does not. If it works we have succeeded, otherwise it would be put aside.

2-Here,What we do is to collect all the variables into a single float variable f . In fact, it is Some sort of variable change or transformation.

3- Any way u and v , some how are related to each other, in advance we do not know this relationship, one possibility is that, they both are the function of the same function. this is the case, in many physical situations, for example in electromagnetic radiation all components of electrical and magnetic fields all are functions of $K.r + \omega t$ and in harmonic oscillation the oscillations in all directions are all functions of ωt .Therefore it is reasonable to think that the two components of the velocity are the functions of a common function like $f(x, t)$.

Now according to the above considerations equation(3) will be written as;

$$\frac{du}{df} \frac{\partial f}{\partial x} + \frac{dv}{df} \frac{\partial f}{\partial y} = 0 \quad (5)$$

Since f is a float and arbitrary function, so far as it fulfills the conditions expressed by the equations 1- 4,any properties can be attributed to that. Obviously at first we prefer to chose f in such a way that the equations could be solved as easily as possible. We do have so Many options and as the first choice we assume

$$\beta \frac{\partial f}{\partial x} = \alpha \frac{\partial f}{\partial y} \quad (6)$$

with this assumption equation(5) will be simplified to

$$\alpha \frac{du}{df} + \beta \frac{dv}{df} = 0 \quad (7)$$

which immediately results in

$$\alpha u(f) + \beta v(f) = c \quad (8)$$

As we mentioned this is not the only possible choice and for example we may choose f to be

$$\frac{\partial f}{\partial x} = f \frac{\partial f}{\partial y} \quad (9)$$

or

$$\frac{\partial f}{\partial x} = e^f \frac{\partial f}{\partial y} \quad (10)$$

What choice is the most suitable one, it depends on the boundary conditions that are to be satisfied, at this point let us concentrate on equations (6) and (7) and look for the consequences. From elementary algebra The solution to equation(6) is ,

$$f = f(\alpha x + \beta y) \quad (11)$$

That is to say f can be any arbitrary function of $(\alpha x + \beta y)$. From equations(7) and (8) we have

$$\frac{\partial v}{\partial x} = -\frac{\alpha}{\beta} \frac{\partial u}{\partial x}, \quad \frac{\partial^2 v}{\partial x^2} = -\frac{\alpha}{\beta} \frac{\partial^2 u}{\partial x^2} \quad (12)$$

Substituting results of (11) into equation(2), we will come to

$$u(x, y) \frac{\partial u(x, y)}{\partial x} + v(x, y) \frac{\partial u(x, y)}{\partial y} = \frac{\beta}{\alpha} \frac{1}{\rho} \frac{\partial}{\partial y} P(x, y) + \nu \left(\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} \right) \quad (13)$$

subtracting equation(1) from equation(12) will result in

$$\frac{1}{\rho} \left(\alpha \frac{\partial P}{\partial x} + \beta \frac{\partial P}{\partial y} \right) = 0 \quad (14)$$

The general solution to this equation is $P = P(\beta x - \alpha y)$, that is, P can be any function of $(\beta x - \alpha y)$.

Now if we rewrite equation(1) in this way;

$$u(x, y) \frac{\partial u(x, y)}{\partial x} + v(x, y) \frac{\partial u(x, y)}{\partial y} - \nu \left(\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} \right) = -\frac{1}{\rho} \frac{\partial}{\partial x} P(x, y) \quad (15)$$

it will be seen that, while according to our first assumption the left hand side of the above equation is a function of $f(\alpha x + \beta y)$ the right hand side has to be a function of $(\beta x - \alpha y)$, but these functions are linearly independent. This can not happen unless $\frac{\partial P}{\partial x} = 0$ or $\frac{\partial p}{\partial x} = \text{constant}$. The same argument is true for y dependence of P ; that is, $\frac{\partial P}{\partial y} = 0$ or $\frac{\partial P}{\partial y} = \text{constant}$. In this way we find that our original assumption $u = u(f)$ and $v = v(f)$ is only consistent with those physical situations, at which

$$p = \text{constant} \quad (16)$$

or

$$p = \beta x - \alpha y \quad (17)$$

So far we have derived two answer for P , and each will lead to a different result. If we follow our analysis by taking into account the result of equation (15) and substituting that into the equations (1) and (2), we will see that both of these equations are identical. Therefore we need only one of them to be solved. As a result equation(1) will be written as

$$u(x, y) \frac{\partial u(x, y)}{\partial x} + v(x, y) \frac{\partial u(x, y)}{\partial y} = \nu \left(\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} \right) \quad (18)$$

since $u = u(f)$, the following results are valid

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} \frac{du}{df} \quad \frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} \frac{du}{df} \quad (19)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 u}{df^2} \left(\frac{\partial f}{\partial x} \right)^2 + \frac{\partial^2 f}{\partial x^2} \frac{du}{df} \quad \frac{\partial^2 u}{\partial y^2} = \frac{d^2 u}{df^2} \left(\frac{\partial f}{\partial y} \right)^2 + \frac{\partial^2 f}{\partial y^2} \frac{du}{df} \quad (20)$$

From equations(6), (8), (19) and (20) will will have

$$\frac{c}{\beta} \frac{du}{df} \frac{\partial f}{\partial y} = \nu \left(\frac{d^2 u}{df^2} \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] + \frac{du}{df} \left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right] \right) \quad (21)$$

Up to this point u has been separated from other parameters. To be able to solve for u we have to decide about f . In the next two subsections we will try two different cases.

2.1 $f = \alpha x + \beta y$

As we said f can be any function of $\alpha x + \beta y$ and each f will give results that are consistent with an especial boundary conditions. The simplest choice will be

$$f = \alpha x + \beta y \quad (22)$$

For this choice, equation(20) will be reduced to

$$\frac{c}{\nu(\alpha^2 + \beta^2)} \frac{d^2 u}{df^2} = \frac{d^2 u}{df^2} \quad (23)$$

The obvious solution to this differential equation is

$$u = Ae^{\frac{c}{\nu(\alpha^2+\beta^2)}(\alpha x+\beta y)} + B \quad (24)$$

where A and B are constants of integration. Substituting the above result into the equation(8) v will be found out to be;

$$v(f) = \frac{1}{\beta}(c - \alpha B - \alpha Ae^{\frac{c}{\nu(\alpha^2+\beta^2)}(\alpha x+\beta y)}) \quad (25)$$

2.2 $f = e^{\alpha x+\beta y}$

For this f we will have;

$$\frac{\partial f}{\partial x} = \alpha f, \quad \frac{\partial^2 f}{\partial x^2} = \alpha^2 f, \quad \frac{\partial f}{\partial y} = \beta f, \quad \frac{\partial^2 f}{\partial y^2} = \beta^2 f \quad (26)$$

consequently equation(20) will be

$$\left(\frac{c}{\nu(\alpha^2+\beta^2)} - 1\right)\frac{du}{df} = f\frac{d^2u}{df^2} \quad (27)$$

which has the solution

$$u = A + Bf^{\frac{c}{\nu(\alpha^2+\beta^2)}} \quad (28)$$

or in terms of x and y it will be

$$u = A + B(\alpha x + \beta y)^{\frac{c}{\nu(\alpha^2+\beta^2)}} \quad (29)$$

Accordingly from equation(8) v will be

$$v = \frac{c - \alpha A}{\beta} - \alpha(\alpha x + \beta y)^{\frac{c}{\nu(\alpha^2+\beta^2)}} \quad (30)$$

Of course for each choice of f we will have a different solution. But the solutions found in this way all have the simple dependance on $\alpha x + \beta y$. Now we are looking for solution that will have more complicated dependance on x and y . This helps us to find solutions that are consistent with other type of boundary conditions. In the next section we will study this aspect of the problem.

3 Other solutions

In this section the basic idea is that, having two different particular solution, how we may find other solutions. Suppose we select the two following particular solutions .

$$u_1 = Ae^{\frac{c}{\nu(\alpha^2+\beta^2)}(\alpha x+\beta y)} + B \quad (31)$$

Of course if we substitute $-\beta$ in place of β in the above equation the result will also be another linear independent solution to our equations. Therefore our next choice will be.

$$u_2 = Ae^{\frac{c}{\nu(\alpha^2+\beta^2)}(\alpha x-\beta y)} + B \quad (32)$$

We have to remind that we are quite free to chose any particular solution we like.

Now we look for those u and v and P that are functions of u_1 and u_2 . That is to say :

$$u = u(u_1, u_2) \quad , v = v(u_1, u_2), \quad P = P(u_1, u_2), \quad (33)$$

Substituting equations(31) in the continuity equation, (equation(3)) will lead to:

$$\frac{\partial u}{\partial u_1} \frac{\partial u_1}{\partial x} + \frac{\partial u}{\partial u_2} \frac{\partial u_2}{\partial x} + \frac{\partial v}{\partial u_1} \frac{\partial u_1}{\partial y} + \frac{\partial v}{\partial u_2} \frac{\partial u_2}{\partial y} = 0 \quad (34)$$

If we differentiate u_1 and u_2 with respect to x and y and if we substitute the result in equation(32) we will have

$$e^{\frac{c}{\nu(\alpha^2+\beta^2)}(\alpha x+\beta y)} (\alpha \frac{\partial u}{\partial u_1} + \beta \frac{\partial v}{\partial u_1}) + e^{\frac{c}{\nu(\alpha^2+\beta^2)}(\alpha x-\beta y)} (\alpha \frac{\partial u}{\partial u_2} - \beta \frac{\partial v}{\partial u_2}) = 0 \quad (35)$$

Since the exponentials are linearly independent, it is obvious that we may write

$$\alpha \frac{\partial u}{\partial u_1} + \beta \frac{\partial v}{\partial u_1} = 0 \quad (36)$$

$$\alpha \frac{\partial u}{\partial u_2} - \beta \frac{\partial v}{\partial u_2} = 0 \quad (37)$$

Upon integration we will come to

$$\alpha u + \beta v = G(u_2) \quad (38)$$

$$\alpha u - \beta v = H(u_1) \quad (39)$$

Where G and H are arbitrary functions. From the above equations we will come to

$$u = \frac{1}{2\alpha}(G(u_2) + H(u_1)) \quad (40)$$

$$v = \frac{1}{2\beta}(G(u_2) - H(u_1)) \quad (41)$$

As before, depending on what functions we choose for G and H , we will come to different results, Suppose we choose $G = u_2$ and $H = u_1$ therefore

$$u = \frac{1}{2\alpha}(u_1 + u_2) \quad (42)$$

$$v = \frac{1}{2\beta}(u_1 - u_2) \quad (43)$$

Substituting the above results in equations (1) and (2) we will come to the following results for $P(x, y)$

$$\frac{\frac{\partial}{\partial x} P(x, y)}{\rho} = 1/2 \left(-2 A e^{\frac{c(\alpha x + \beta y)}{\nu(\alpha^2 + \beta^2)}} + \left(1 + e^{2 \frac{c\beta y}{\nu(\alpha^2 + \beta^2)}} \right) (c - B) \right) A e^{\frac{c(\alpha x - \beta y)}{\nu(\alpha^2 + \beta^2)}} c \alpha^{-1} \nu^{-1} (\alpha^2 + \beta^2)^{-1} \quad (44)$$

$$\frac{\frac{\partial}{\partial y} p(x, y)}{\rho} = 1/2 (-c + B) A c e^{\frac{c(\alpha x - \beta y)}{\nu(\alpha^2 + \beta^2)}} \left(-1 + e^{2 \frac{c\beta y}{\nu(\alpha^2 + \beta^2)}} \right) \beta^{-1} \nu^{-1} (\alpha^2 + \beta^2)^{-1} \quad (45)$$

If we take $B = c$ we will have

$$\frac{\partial}{\partial y} p = 0 \quad (46)$$

and

$$\frac{\partial}{\partial} p = -\rho A^2 e^{2 \frac{c\alpha x}{\nu(\alpha^2 + \beta^2)}} c \alpha^{-1} \nu^{-1} (\alpha^2 + \beta^2)^{-1} \quad (47)$$

or

$$p = -1/2 A^2 e^{2 \frac{c\alpha x}{\nu(\alpha^2 + \beta^2)}} \alpha^{-2} + D \quad (48)$$

Where D is a constant of integration.

As a result we may say that the set of functions u , v and P defined by equations(29),(30),(40),(41),(46) are new generations of the solutions to Navier stokes equations.

4 Discussion

The retrieved results show that there are an infinite number of solutions to the Navier stokes equations. Any choice for $f(\alpha x + \beta)y$ in equation(20) will result in a new solution. How to chose f , depends on the Boundary conditions to be applied.

Since all f 's in section 2 were all specifically functions of $\alpha x + \beta y$ the resulting solutions for u and v , will also be only dependent on $\alpha x + \beta y$. Therefore they will, be consistent to some especial set of boundary conditions. In section 3, In order to increase the capabilities of the solutions, we tried to find out solutions with different type of dependency on x and y . Mathematically ,what we did in that section is somehow similar to what we do in linear scheme. In linear scheme we prove that if u_1 and u_2 are particular solutions of a linear differential equation, then $u = c_1u_1 + c_2u_2$ will be the general solution for that differential equation. In a similar manner, in section 3, we showed that if u_1 and u_2 are particular solutions to Navier stokes equations then, $u = \frac{1}{2\alpha}(G(u_2) + H(u_1))$ will be a general solution to that equation. Since from the results of section 2 there are numerous Particular solutions, available, therefore numerous set of general solutions are predictable for our set of nonlinear partial differential equations.

The results found this way in nonlinear regime is rather different from what we come to, in linear discussion. In linear discussion, We argue that if, u_1 and u_2 are particular solutions, then the only general solution will be $u = c_1u_1 + c_2u_2$, while here in the recent discussion we come to the result that for any pair of particular solutions we will have a new set of general solutions. consequently one can not recognise from the beginning which set of General solutions are consistent with the required boundary condition.

To argue this point, is beyond the scope of this article. Here, What we may suggest is that; It is possible to focus from the beginning, on those float functions f that are consistent with all or part of the boundary conditions and continue our calculations based on those choices.

In short,The scheme we followed in this article provides us numerous acceptable solutions, and it shows that with preliminary knowledge of calculus analysis we can overcome the complexities of solving nonlinear set of partial differential equations.

At the end, what we suggest as the topics of studies in future is; "how we may consider the effects of boundary conditions from the beginning".

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